EXTENSIONS BY SPACES OF CONTINUOUS FUNCTIONS

JESÚS M. F. CASTILLO AND YOLANDA MORENO

Abstract. We characterize the Banach spaces X such that $\operatorname{Ext}(X, C(K)) = 0$ for every compact space K.

1. Introduction

In this paper we give a necessary and sufficient condition on a Banach space X so that every The exact sequence

$$0 \longrightarrow C(K) \longrightarrow E \longrightarrow X \longrightarrow 0,$$

splits. Such exact sequences are sometimes called *extensions of* X *by* $C(K)$. Thus, we shall characterize those Banach spaces X such that, for every compact Hausdorff space K one has $\operatorname{Ext}(X, C(K)) = 0$. The characterization is given in terms of properties of the metric projection from the space $Z(X, \mathbb{R})$ of $\mathbb{R}$-valued z-linear maps defined on X onto the algebraic dual X'.

Previous results in this direction were obtained by Kalton and Pełczyński (see [11]) who had proved that if $\operatorname{Ext}(X, C(K)) = 0$ then X must have the Schur property; then Kalton shows in [9] that if $\operatorname{Ext}(X, C(K)) = 0$ then X must also have the strong-Schur property. In [6], Johnson and Zippin show that $\operatorname{Ext}(X, C(K)) = 0$ for every space X dual of a subspace of c_0, while Kalton proves in [9] that the converse is also true when X has an unconditional FDD. The paper [3] gives the first steps towards a theory of extensions with $C(K)$-spaces by establishing conditions under which $\operatorname{Ext}(X, C[0,1]) \neq 0$ or $\operatorname{Ext}(X, C(\omega^\omega)) \neq 0$.

2. Background on the z-linear representation of extensions

An exact sequence of Banach spaces is a diagram $0 \to Y \xrightarrow{j} E \xrightarrow{q} X \to 0$ composed with Banach spaces and operators in such a way that the kernel of each arrow coincides with the image of the preceding. It is sometimes called an extension of X by Y or, simply, an extension of X. The open mapping theorem makes Y a subspace of E through the embedding j and X the corresponding quotient space through q.

Supported in part by DGICYT project 2001-0813.
2000 Math. Subject Class. 46B03, 46B07.

Following the theory developed by Kalton [7] and Kalton and Peck [10] in which extensions $0 \to Y \to E \to X \to 0$ of quasi-Banach spaces were identified with quasi-linear maps $F : X \to Y$, extensions of Banach spaces can be represented (see [5, 2]) with a particular type of quasi-linear maps called z-linear maps. These are homogeneous maps such that for some constant C and every finite set of points $\{z_1, \ldots, z_n\} \subset X$ one has

$$\|F(\sum_{i=1}^{n} z_i) - \sum_{i=1}^{n} F(z_i)\| \leq C \sum_{i=1}^{n} \|z_i\|.$$

The infimum of the constants C above is denoted $Z(F)$.

We need to introduce some background about z-linear maps since most of our work shall be developed working with this representation of exact sequences. A z-linear map $F : X \to Y$ determines a quasi-norm on the product space $Y \times X$ given by $\|(y, x)\|_F = \|y - Fx\| + \|x\|$. Let us call this quasi-Banach space $Y \oplus_F X$. If $co(Y \oplus_F X)$ denotes its Banach envelope (i.e., the Banach space having as unit ball the closed convex hull of the unit ball of $\| \cdot \|_F$, it is easy to see that $co(Y \oplus_F X)$ is $Z(F)$-isomorphic to $Y \oplus_F X$. In this way each z-linear map induces an exact sequence $0 \to Y \stackrel{j_F}{\to} co(Y \oplus_F X) \stackrel{q_F}{\to} X \to 0$ of Banach spaces with embedding $j_F(y) = (y, 0)$ and quotient map $q_F(y, x) = x$. To obtain a z-linear map associated to a given exact sequence $0 \to Y \to E \to X \to 0$ of Banach spaces one can proceed as follows: take an homogeneous and bounded selection $b : X \to E$ for the quotient map, then take a linear selection $l : X \to E$ for the quotient map, and finally make the difference $F = b - l$, which is a z-linear map $X \to Y$. The exact sequences $0 \to Y \to E \to X \to 0$ and $0 \to Y \to co(Y \oplus_F X) \to X \to 0$ are *equivalent*, in the sense that there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle T} & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & co(Y \oplus_F X) & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

The classical 3-lemma from homological algebra plus the open mapping theorem imply that T is an isomorphism. With this meaning we shall say that the z-linear map F has been associated to the exact sequence and we will represent this using the notation $0 \to Y \stackrel{j}{\to} E \stackrel{q}{\to} X \to 0 \equiv F$.

Two z-linear maps F and G are said to be *equivalent*, and written $F \equiv G$, when they induce equivalent exact sequences. The classical theory establishes that this happens if and only if there is a linear map $L : X \to Y$ such that $\|F - G - L\| = \sup\{\|(F - G - L)(x)\| : \|x\| \leq 1\} < +\infty$. We will say sometimes that G is a version of F. An exact sequence $0 \to Y \stackrel{j}{\to} E \stackrel{q}{\to} X \to 0 \equiv F$ is said to split if it is equivalent to the *trivial* sequence $0 \to Y \to Y \oplus X \to X \to 0$; equivalently, $j(Y)$ is complemented in E. In z-linear terms this means $F \equiv 0$, namely, there is a

linear map $L : X \to Y$ such that $\|F - L\| < +\infty$. We write $\mathrm{Ext}(X, Y) = 0$ to mean that every exact sequence with Y as subspace and X as quotient splits. The lower sequence in a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \longrightarrow & Z & \longrightarrow & 0 \equiv F \\ & & {\scriptstyle T}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

is called the *push-out sequence* and has TF as associated z-linear map. One has that $TF \equiv 0$ if and only if T extends to X through j. Dually, the lower sequence in a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F \\ & & \|\uparrow & & \uparrow{\scriptstyle S} & & & & \\ 0 & \longrightarrow & Y & \longrightarrow & X' & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

is called the *pull-back sequence* and has FS as associated z-linear map. One has that $FS \equiv 0$ if and only if S can be lifted to to X through q.

3. Linearization and factorization of z-linear maps

It will be necessary for us to deal with the space $Z(X, Y)$ of z-linear maps $F : X \to Y$ considered as mere functions (i.e., without equivalence relation). The space $Z(X, \mathbb{R})$ admits a semi-normed (not necessarily Hausdorff) topology induced by the seminorm $Z(\cdot)$ (the constant of z-linearity).

In order to get our main result, Theorem 1, we need a refinement of one of the main results in [3, Thm. 2.1]. There it was shown that for each separable space X there exists a universal exact sequence $0 \to C[0, 1] \to E \to X \to 0 \equiv U$ with the property that each exact sequence $0 \to C[0, 1] \to E_1 \to X \to 0 \equiv F$ is a push-out of U. In particular, there exists an operator $\phi : C[0, 1] \to C[0, 1]$ such that $F \equiv \phi U$. Our aim is to get an equality in the previous result instead of the mere equivalence. With that purpose in mind we introduce the linearization and factorization processes, and recall Zippin's extension method.

Linearization process. Given a z-linear map $F : X \to Y$ and a Hamel basis (e_γ) for Z, we define a linear map $\ell_F : X \to Y$ by setting $\ell_F(e_\gamma) = F(e_\gamma)$. The process $F \to \ell_F$ is linear. The *linearized form* of F (with respect to a given Hamel basis) is its version $F - \ell_F$. We shall call the correspondence $L : Z(X, Y) \to Z(X, Y)$ given by $L(F) = F - \ell_F$ *linearization process* (with respect to a given Hamel basis). We shall omit from now own the coda "with respect to a given Hamel basis".

Factorization process. Consider a minimal (in the purely algebraic sense) set S^+ such that the unit sphere S of X coincides with $\cup_{z \in S^+} \{z, -z\}$. Naming $(e_z)_{z \in S^+}$ the

canonical basis of $l_1(S^+)$, we construct the quotient map $q : l_1(S^+) \to Z$ defined by $q(e_z) = z$. An homogeneous bounded section for q comes defined as: If $p \in S^+$ then $b(p) = e_p$; the map b can be extended by homogeneity to S and then to the whole X. The z-linear map $P = b - \ell_b$ is associated with the exact sequence $0 \to K(X) \to l_1(S^+) \xrightarrow{q} X \to 0$, which we call projective presentation of X. Given a z-linear map $F : X \to Y$ we construct the associated sequence $0 \to Y \xrightarrow{j_F} co(Y \oplus_F X) \xrightarrow{q_F} X \to 0 \equiv F$. A homogeneous bounded selection for q_F is $B(x) = (Fx, x)$; it has norm 1 since $\|Bx\| = \|x\|_F \leq \|x\|$. A linear selection for q_F is $\Upsilon(x) = (0, x)$ and one has $F = B - \Upsilon$. We define a norm one operator $\phi(F) : l_1(S^+) \to co(Y \oplus_F X)$ as:

$$\phi(F)(\sum_{z\in S^+} \lambda_z e_z) = \sum_{z\in S^+} \lambda_z Bqe_z.$$

The restriction of this operator to $K(X)$ shall be called ϕ_F, and it takes values in Y. Since the norm of Y is $Z(F)$-equivalent to that of $co(Y \oplus_F X)$ we get $\|\phi_F : K(X) \to Y\| \leq Z(F)$. We call *factorization process* the correspondence $Z(X, Y) \to \mathcal{L}(K(X), Y)$ given by $F \to \phi_F$. Observe now that $B = \phi(F)b$ because if $p \in S^+$ then $\phi(F)b(p) = \phi(F)(e_p) = Bqe_p = Bp$. Hence

$$\phi_F P = \phi(F)b - \phi(F)\ell_b = B - \phi(F)\ell_b = B - \Upsilon + \Upsilon - \phi(F)\ell_b = F + \Upsilon - \phi(F)\ell_b.$$

Since P is linearized form so must be $\phi(F)P$ and therefore $\Upsilon - \phi(F)\ell_b = -\ell_F$. This means that when the factorization process is applied to a z-linear map in linearized form $G = F - \ell_F$ then one obtains $\phi_G P = G$.

Zippin's extension method. Let $\delta_X : X \to C(B_{X^*}, w^*)$ be the canonical embedding. As it was observed by Zippin in [14, 15], the w^*-continuous map $\tau : B_{X^*} \to B_{C(B_{X^*})^*}$ defined by $\tau(x^*)(f) = f(x^*)$ provides an extension for every norm-one operator $T : X \to C(K)$ to $C(B_{X^*})$ through δ_X in the following way: $\widehat{T}(f)(k) = \tau(T^*(\delta_k))(f)$. We will say that $\widehat{T}$ is the Zippin extension of T.

Putting together the three processes we get.

Lemma 1. *For every Banach space X there exists a compact space $\Xi[X]$ and a z-linear map $\Delta : X \to C(\Xi[X])$ such that for every z-linear map $F : X \to C(K)$ in linearized form there exists an operator $Z(F)\Phi_F : C(\Xi[X]) \to C(K)$ with norm $\|\Phi_F\| \leq Z(F)$ such that $Z(F)\Phi_F\Delta = F$.*

Proof. Consider the extension $0 \to K(X) \to l_1(S^+) \to X \to 0 \equiv P$ and the operator ϕ_F given by the factorization process as described above. Let Φ_F be Zippin's norm one extension of $Z(F)^{-1}\phi_F$. It is clear that $Z(F)\Phi_F\Delta = Z(F)\Phi_F\delta_X P = Z(F)Z(F)^{-1}\phi_F P = F$. □

4. Characterization of the spaces X such that $\mathrm{Ext}(X, C(K)) = 0$

Besides the semi-normed topology, we will need to consider on the space $Z(X, \mathbb{R})$ the topology w^* of pointwise convergence: we shall say that $F = w^* - \lim F_\alpha$ if for every $x \in X$ one has $F(x) = \lim F_\alpha(x)$. A map $Z(X, \mathbb{R}) \to Z(X, \mathbb{R})$ will be called w^*-continuous if it is w^*-continuous on the unit ball; namely, it transforms w^*-convergent nets on the unit ball into w^*-convergent nets. If X' denotes the algebraic dual of X, then $X' \subset Z(X, \mathbb{R})$ and the restriction of the w^*-topology to X' is the weak $w(X', X)$-topology.

The so-called "nonlinear Hahn-Banach theorem" shown in [2] asserts that given a z-linear map $F : X \to \mathbb{R}$ there exists a linear map $L \in X'$ such that $\|F-L\| \leq Z(F)$. This makes non-vacuous the following definition:
Definitions. A map $m : Z(X, \mathbb{R}) \to X'$ will be called a λ-metric projection if

$$\|F - m(F)\| \leq \lambda Z(F).$$

If we do not need to emphasize λ we just speak of metric projection. We shall say that a Banach space X admits a metric projection with a given property P if there is a metric projection $m : Z(X, \mathbb{R}) \to X'$ with property P.

For instance, every Banach space admits a 1-metric projection. In [4] it was shown that a Banach space X is an $\mathcal{L}_1$-space if and only if it admits a *linear* metric projection, which This yields that X admits a *linear* metric projection if and only if $\mathrm{Ext}(X, Y^*) = 0$ for every dual space. A characterization of Banach spaces such that all their extensions by any $C(K)$-space are trivial can also be obtained in terms of properties of metric projections.

Theorem 1. *A Banach space X admits a w^*-continuous metric projection if and only if $\mathrm{Ext}(X, C(K)) = 0$ for every compact Hausdorff space K.*

Proof of the necessity. Let $F : X \to C(K)$ be a z-linear map and let $m : Z(X, \mathbb{R}) \to X'$ be a w^*-continuous λ-metric projection. We define a linear map $M : X \to C(K)$ by

$$M(x)(k) = m(\delta_k F)(x),$$

where δ_k is the evaluation at k. The map M is well defined since $M(x)$ is a continuous function: whenever $k = \lim k_\alpha$ on K then $M(x)(k) = m(\delta_k F)(x) = m(w^* - \lim \delta_{k_\alpha} F)(x) = \lim m\left(\delta_{k_\alpha} F\right)(x) = \lim M(x)(k_\alpha)$. Moreover, $|F(x)(k) - M(x)(k)| = |\delta_k F(x) - m(\delta_k F)(x)| \leq \lambda Z(\delta_k F)\|x\|$, which implies $\|F - M\| \leq \lambda Z(F)$.

Proof of the sufficiency . If $\mathrm{Ext}(X, C(K)) = 0$ for every $C(K)$-space then $\mathrm{Ext}(X, C(\Xi[X])) = 0$. In particular, $\Delta \equiv 0$ (the "initial" z-linear map of Lemma 1) and there exists a linear map $\Lambda : X \to C(\Xi[X])$ such that $\|\Delta - \Lambda\| < +\infty$. Let $F : X \to \mathbb{R}$ be a z-linear map with $Z(F) \leq 1$. Consider the projective presentation $0 \to K(X) \to l_1(S^+) \to X \to 0 \equiv P$ and the operator $\phi_F : K(X) \to \mathbb{R}$ such

that $\phi_F P = F - \ell_F$ given by the factorization process. Let $\Phi_F : C(\Xi[X]) \to \mathbb{R}$ be Zippin's extension. We define a map $m : Z(X, \mathbb{R}) \to X'$ as

$$m(F) = \Phi_F \Lambda.$$

To prove that $m(\cdot)$ is w^*-continuous we decompose it in three applications:

(1) The linearization process $L(F) = F - \ell_F$, which is w^*-continuous;

(2) The factorization process. We show now it is w*-continuous. Observe that if we restrict ourselves to work with the subspace $\varphi_0(S^+)$ of $l_1(S^+)$ of all finitely supported sequences then $K_0 = K(X) \cap \varphi_0(S^+)$ is dense in $K(X)$. On this dense subspace the operator ϕ_F takes the form

$$\phi_F(\sum \lambda_z e_z) = \sum \lambda_z F q e_z.$$

Tus, if $\{G_\alpha\}$ is a net in the unit ball such that $G = w^* - \lim G_\alpha$ then $\phi_G(u) = \lim \phi_{G_\alpha(u)}$ for all $u \in K_0$. Since $\|\phi_{G_\alpha}\| \leq 1$, the sequence of uniformly bounded operators is w^*-convergent on the whole $K(X)$.

(3) Zippin's extension method is a w^*-continuous process $B_{K(X)^*} \to B_{C(\Xi[X])^*}$ as we have already remarked.

The w^*-continuous metric projection we are looking for is

$$F \to m(F) + \ell_F.$$

It is obviously w^*-continuous and it remains to show that it is a metric projection:

$$\|F - m(F) - \ell_F\| = \|\Phi_F \Delta - \Phi_F \Lambda\| \leq \|\phi_F\| \|\Delta - \Lambda\| \leq Z(F) \|\Delta - \Lambda\|.$$

Remark. In quantitative terms the previous arguments provide that every $C(K)$-valued z-linear map admits a linear map at distance at most $\lambda Z(\cdot)$ if and only if there is a w^*-continuous method to assign to each $\mathbb{R}$-valued z-linear map a linear map at distance at most $\lambda Z(\cdot)$.

5. Remarks and open questions

1. *Existence of w^*-continuous metric projections.* In general, every Banach space X admits both 1-metric projections (such as that given by the non-linear Hahn-Banach theorem) and w^*-continuous projections (such as the linear map $F \to \ell_F$). As we have already seen, it is extremely unusual that the former are w^*-continuous and the latter is (usually) not a metric projection.

2. *Existence of w^*-continuous C-metric projections on finite dimensional spaces.* There is one instance in which the the linear map $F \to \ell_F$ is a metric projection: when the space is l_1, we work on the dense subspace of finitely supported sequences and take as Hamel basis the canonical Schauder basis of l_1. The final step of passing from a dense subspace to the whole space is a classical extension result for z-linear maps (see [10, 5]). It is therefore clear that finite-dimensional spaces E admit a w^*-continuous $\mathrm{dist}(E, l_1^{dimE})$-metric projection. However, as the dimension of E increases the "λ-metric" character of the projection is spoiled. Nevertheless, it is

possible to get w^*-continuous λ-metric projections on finite dimensional spaces with uniform constant λ independent of the dimension. To do that observe that $\mathcal{L}_{\infty,\mu}$-spaces are *locally complemented* in any superspace, which means (see [8]) that there exists a constant $C(\mu)$ such that for every finite-dimensional space E every z-linear map $F : E \to \mathcal{L}_{\infty,\mu}$ admits a linear map $\ell : E \to \mathcal{L}_{\infty,\mu}$ such that $\|F - \ell\| \leq C(\mu)$. Since $C(K)$-spaces are $\mathcal{L}_{\infty,1+\varepsilon}$-spaces, it turns out that finite dimensional spaces admit w^*-continuous $C(1+\varepsilon)$-metric projections.

3. *Existence of w^*-continuous metric projections in $l_1(X_n)$.* To work with infinite dimensional spaces we recall again the extension result for z-linear maps; thus, we only need to work on a dense subspace; i.e., if X_0 denotes a dense subspace of X, then X_0 admits a w^*-continuous metric projection if and only if X does.

Proposition 1. *If A_n is a sequence of spaces admitting w^*-continuous λ-metric projections then the vector sum $l_1(A_n)$ admits a w^*-continuous $(1+\lambda)$-metric projection.*

Proof. Let $\varphi_0(A_n)$ be the dense subspace of $l_1(A_n)$ formed by all finitely supported sequences. The map $Z(\varphi_0(A_n), \mathbb{R}) \to l_\infty(Z(A_n, \mathbb{R}))$ that assigns to a z-linear map F the sequence $(F_{|A_n})_n$ of its restrictions is clearly w^*-continuous. If $m_n : Z(A_n, \mathbb{R}) \to A_n'$ are w^*-continuous metric projections then we can define a w^*-continuous metric projection $m : Z(\varphi_0(A_n), \mathbb{R}) \to l_1(A_n)'$ as follows: take $(e^n_\gamma)_\gamma$ a Hamel basis for A_n, and set $m(F)(e^n_\gamma) = m_n(F_n)(e^n_\gamma)$. If $a = (a_n) \in \varphi_0(A_n)$ then

$$|Fa - m(F)a| = |Fa - \sum F_n a_n + \sum F_n a_n - \sum m_n(F_n) a_n| \leq (1+\lambda) Z(F) \sum \|a_n\|.$$

□

4. *The role of the $C(K)$-space.* It is not difficult to see that for separable spaces X the condition "$\mathrm{Ext}(X, C(K)) = 0$ for all compact spaces K" is equivalent to "$\mathrm{Ext}(X, C[0,1]) = 0$". It has been shown in [3] that $\mathrm{Ext}(X, C(\omega^\omega)) = 0$ is a strictly weaker condition; precisely, if T denotes the dual of the original Tsirelson space then $\mathrm{Ext}(T, C[0,1]) \neq 0$ while $\mathrm{Ext}(T, C(\omega^\omega)) = 0$. It seems a touchy question how metric projections would reflect the change of the $C(K)$ target space. For instance, it is clear that, without separability assumptions, if X admits a metric projection which is w^*-continuous at 0 then $\mathrm{Ext}(X, c_0) = 0$.

Question. Is it true that $\mathrm{Ext}(X, c_0) = 0$ if and only if X admits a metric projection w^*-continuous at 0? In particular: Does every separable Banach space admit a metric projection w^*-continuous at 0?

6. *On the equation* $\mathrm{Ext}(X, C(K)) = 0$ *for subspaces of $l_1(\Gamma)$.* Returning to Kalton's proposition that separable Banach spaces such that $\mathrm{Ext}(X, C[0,1]) = 0$ must have the strong-Schur property, it makes sense to approach the converse by asking:

Question. Is $\mathrm{Ext}(M, C[0,1]) = 0$ for every subspace M of l_1?

Let us show the existence of a subspace $K(Z)$ of $l_1(\Gamma)$ such that $\mathrm{Ext}(K(Z), c_0) \neq 0$, which implies that $K(Z)$ does not admit a metric projection w^*-continuous at 0. To get the example, recall that l_∞/c_0 is not injective (see [13]) and thus there exists some (necessarily nonseparable) space Z for which $\mathrm{Ext}(Z, l_\infty/c_0) \neq 0$. This gives a nontrivial exact sequence $0 \to l_\infty/c_0 \to E \to Z \to 0 \equiv F$. Given an exact sequence $0 \to K(Z) \xrightarrow{i} l_1(\Gamma) \to Z \to 0 \equiv P$ there is an operator $\phi : K(Z) \to l_\infty/c_0$ such that $F \equiv \phi P$. Consider the exact sequence $0 \to c_0 \to l_\infty \xrightarrow{q} l_\infty/c_0 \to 0 \equiv I$. If $I\phi \equiv 0$ then ϕ can be lifted to an operator $\eta : K(Z) \to l_\infty$ through q; this operator η, in turn can be extended to an operator $\psi : l_1(\Gamma) \to l_\infty$ through i. The operator $q\psi : l_1(\Gamma) \to l_\infty/c_0$ extends ϕ since $q\psi i = q\eta = \phi$. Therefore $F \equiv \phi P \equiv 0$, against the hypothesis.

We close the paper observing that a remarkable class of spaces with the strong-Schur property are those having a l_1-skipped blocking decomposition introduced by Bourgain and Rosenthal [1]. It would be interesting to know if those spaces admit w^*-continuous metric projections.

References

[1] J. Bourgain and H. P. Rosenthal, *Geometrical implications of certain finite dimensional decompositions,* Bull. Math. Soc. Belgium 32 (1980) 57-82.

[2] F. Cabello Sánchez and J.M.F. Castillo, *Duality and twisted sums of Banach spaces,* J. Funct. Anal. 175 (2000) 1-16.

[3] F. Cabello Sánchez, J.M.F. Castillo, N. J. Kalton and D. Yost, Twisted sums with $C(K)$-spaces, Trans. Amer. Math. Soc. 355 (2003) 4523-4541.

[4] F. Cabello Sánchez, J.M.F. Castillo and F. Sánchez, *Nonlinear metric projections in twisted twilight*, Rev. Real Acad. Ciencias Madrid 94 (2000) 473-483.

[5] J.M.F. Castillo and M. González, *Three-space problems in Banach space theory,* LNM 1667, Springer-Verlag, 1997.

[6] W.B. Johnson and M. Zippin, *Extension of operators from weak*-closed subspaces of l_1 into $C(K)$ spaces*, Studia Math. 117 (1995) 43–55.

[7] N. J. Kalton, *The three-space problem for locally bounded F-spaces*, Compo. Math. 37 (1978) 243–276.

[8] N. Kalton, *Locally Complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$,* Math. Nachr. 115 (1984), 71–97.

[9] N. Kalton, *On subspaces of c_0 and extension of operators into $C(K)$-spaces,* Quart. J. Math. Oxford 52 (2001) 313-328.

[10] N. J. Kalton and N. T. Peck, *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. 255 (1979), 1–30.

[11] N. Kalton, A. Pełczyński, *Kernels of surjections from $\mathcal{L}_1$-spaces with an application to Sidon sets,* Math. Ann. 309 (1997), 135–158.

[12] A. Pełczyński, *Linear extensions, linear averagings and their aplications to linear topological classification of spaces of continuous functions*, Diss. Math. 58 (1968).

[13] H.P. Rosenthal,*On relatively disjoint families of measures, with applications to Banach space theory*, Studia Math. 37 (1970) 13-35.

[14] M. Zippin, *The embedding of Banach spaces into spaces with structure*, Illinois J. Math. 34 (1990) 586 – 606.

[15] M. Zippin, *A global approach to certain operator extension problems*, in LNM 1470, Springer, 1990, 78 – 84.

Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, 06071 Badajoz, Spain

E-mail address: castillo@unex.es

E-mail address: ymoreno@unex.es